ISBN 82-553-0458-4

Mathematics No 11
September 9 1981

DECOMPOSABLE POSITIVE MAPS ON C*-ALGEBRAS

by

Erling Størmer
Dept. of Math., University of Oslo

Decomposable positive maps on C*-algebras

Erling Størmer
Department of Mathematics, University of Oslo

Abstract

It is shown that a positive linear map of a C*-algebra A into $B(H)$ is decomposable if and only if for all $n \in \mathbb{N}$ whenever (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ then $(\phi(x_{ij}))$ belongs to $M_n(B(H))^+$.

A positive linear map ϕ of a C*-algebra A into $B(H)$ - the bounded linear operators on a complex Hilbert sapace H - is said to be decomposable if there are a Hilbert space K, a bounded linear operator v of H into K, and a Jordan homomorphism π of A into $B(K)$ such that $\phi(x)=v^*\pi(x)v$ for all $x \in A$. Such maps have been studied in [2], [3], [5], [7], [8], [9], and are the natural symmetrization of the completely positive ones, defined as those ϕ as above with π a homomorphism. If $M_n(B)$ denotes the $n\times n$ matrices over a subspace B of a C*-algebra and $M_n(B)^+$ the positive part of $M_n(B)$, the celebrated Stinespring theorem [4] states that a map $\phi: A \rightarrow B(H)$ is completely positive if and only if for all $n \in \mathbb{N}$ whenever $(x_{ij}) \in M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$. It is the purpose of the present note to provide an analogous characterization of decomposable maps.

<u>Theorem</u>. Let A be a C^*-algebra and ϕ a linear map of A into $B(H)$. Then ϕ is decomposable if and only if for all $n \in \mathbb{N}$ whenever (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$.

<u>Proof</u>. Suppose ϕ is decomposable, so of the form $v^*\pi v$. If π is a homomorphism (resp. anti-homomorphism) and (x_{ij}) (resp. (x_{ji})) belongs to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$. Since every Jordan homomorphism is the sum of a homomorphism and an anti-homomorphism [6] if both (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ then $(\phi(x_{ij})) \in M_n(B(H))^+$.

Conversely suppose (x_{ij}) and $(x_{ji}) \in M_n(A)^+$ implies $(\phi(x_{ij})) \in M_n(B(H))^+$ for all $n \in \mathbb{N}$. Since this property persists when ϕ is extended to the second dual of A we may assume A is unital and that $A \subset B(L)$ for some Hilbert space L. Let t denote the transpose map on $B(L)$ with respect to some orthonormal basis. Let

$$V = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^t \end{pmatrix} \in M_2(B(L)) : x \in A \right\}.$$

Then V is a self-adjoint subspace of $M_2(B(L))$ containing the identity. Define θ_n on $M_n(B(L))$ by $\theta_n((x_{ij})) = (x^t_{ji})$. Then θ is an anti-automorphism of order 2. Hence if $(x_{ij}) \in M_n(A)$ then $(x_{ji}) \in M_n(A)^+$ if and only if $(x^t_{ij}) = \theta_n((x_{ji})) \in M_n(B(L))^+$. Therefore both (x_{ij}) and (x_{ji}) belong to $M_n(A)^+$ if and only if

$$\left(\begin{pmatrix} x_{ij} & 0 \\ 0 & x^t_{ij} \end{pmatrix} \right) \in M_n(V)^+.$$

Let $\bar{\phi}:V\to B(H)$ be defined by

$$\bar{\phi}\left(\begin{pmatrix} x & 0 \\ 0 & x^t \end{pmatrix}\right) = \phi(x).$$

Then $\bar{\phi}$ is completely positive in the sense of [1] by our hypothesis on ϕ and the above equivalence. By Arveson's extension theorem [1, Thm. 1.2.3] $\bar{\phi}$ has an extension to a completely positive map $\bar{\bar{\phi}}:M_2(B(L))\to B(H)$. By Stinespring's theorem [4] there are a Hilbert space K, a bounded linear map v of H into K, and a representation π_1 of $M_2(B(L))$ on K such that $\bar{\bar{\phi}} = v^*\pi_1 v$. Let π_2 be the Jordan homomorphism of A into $M_2(B(L))$ defined by

$$\pi_2(x) = \begin{pmatrix} x & 0 \\ 0 & x^t \end{pmatrix}, \quad x \in A.$$

Then $\pi=\pi_1\circ\pi_2$ is a Jordan homomorphism of A into $B(K)$ such that $\phi(x)=v^*\pi(x)v$ for all $x \in A$, hence ϕ is decomposable. The proof is complete.

The first example of a nondecomposable positive map was exhibited by Choi [2]. An extension of his example was reproduced in [3] together with a complete proof based on nontrivial results on biquadratic forms. We conclude by giving a short proof of his result. The example is $\phi:M_3(\mathbb{C})\to M_3(\mathbb{C})$ defined by

$$\phi\left(\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}\right) = \begin{pmatrix} \alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_{33} \end{pmatrix} + \mu\begin{pmatrix} \alpha_{33} & 0 & 0 \\ 0 & \alpha_{11} & 0 \\ 0 & 0 & \alpha_{22} \end{pmatrix},$$

where $\mu > 1$. Let $(x_{ij}) \in M_3(M_3(\mathbb{C}))$ be the matrix

$$(x_{ij}) = \left(\begin{array}{ccc|ccc|ccc} \mu & 0 & 0 & 0 & 2\mu & 0 & 0 & 0 & 2\mu \\ 0 & 4\mu^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2\mu & 0 & 0 & 0 & 4\mu & 0 & 0 & 0 & 4\mu \\ 0 & 0 & 0 & 0 & 0 & 8\mu^2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 4\mu^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 2\mu & 0 & 0 & 0 & 4\mu & 0 & 0 & 0 & 4\mu \end{array}\right).$$

Then both (x_{ij}) and (x_{ji}) belong to $M_3(M_3(\mathbb{C}))^+$ while it is easily seen that the matrix $(\phi(x_{ij}))$ is not positive. Hence ϕ is not decomposable by the theorem.

REFERENCES

1. W. Arveson, Subalgebras of C*-algebras, Acta math., 123 (1969), 141-224.

2. M.D. Choi, Positive semidefinite biquadratic forms, Linear Algebra and Appl., 12 (1975), 95-100.

3. M.D. Choi, Some assorted inequalities for positive linear maps on C*-algebras, J.Operator Theory, 4 (1980), 271-285.

4. W.F. Stinespring, Positive functions on C*-algebras, Proc.Amer.Math.Soc., 6 (1955), 211-216.

5. E. Størmer, Positive linear maps of operator algebras, Acta math., 110 (1963), 233-278.

6. E. Størmer, On the Jordan structure of C*-algebras, Trans.Amer.Math.Soc., 120 (1965), 438-447.

7. E. Størmer, Decomposition of positive projections on C*-algebras, Math.Ann., 27 (1980), 21-41.

8. S.L. Woronowicz, Nonextendible positive maps, Commun.Math. Phys., 51 (1976), 243-282.

9. S.L. Woronowicz, Positive maps of low dimensional matrix algebras, Rep. Math.Phys., 10 (1976), 165-183.